

# THE HOMOTOPY CATEGORY OF UNITARY OPERADS AS A FULL SUBCATEGORY OF THE HOMOTOPY CATEGORY OF ALL OPERADS

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**ABSTRACT.** We prove that the homotopy category of topological operads  $\mathcal{P}$  satisfying  $\mathcal{P}(0) = *$  forms a full subcategory of the homotopy category of all topological operads. We more precisely establish that we have a weak-equivalence of simplicial sets at the mapping space level which gives this embedding of homotopy categories when we pass to connected components. We also prove that an analogous result holds for the categories of  $k$ -truncated operads, which are operads defined up to arity  $k$ .

## INTRODUCTION

Let  $\mathcal{O}p$  be the category of all operads in topological spaces. Let  $\mathcal{O}p_* \subset \mathcal{O}p$  be the category formed by the operads such that  $\mathcal{P}(0) = *$  (the category of unitary operads in the terminology of [7]). Recall that this category is isomorphic to the category of  $\Lambda$ -operads  $\Lambda\mathcal{O}p_\emptyset$ , whose objects are operads satisfying  $\mathcal{P}(0) = \emptyset$ , but which we equip with restriction operators  $u^* : \mathcal{P}(n) \rightarrow \mathcal{P}(m)$ , associated to the injective maps  $u : \{1 < \dots < m\} \rightarrow \{1 < \dots < n\}$ , and which model composition operations with an arity zero operation at the inputs  $j \notin \{u(1), \dots, u(m)\}$  (see [7, §I.3]).

Let  $\iota : \mathcal{O}p_* \hookrightarrow \mathcal{O}p$  be the obvious category embedding. This functor has a left adjoint  $\tau : \mathcal{O}p \rightarrow \mathcal{O}p_*$ , which we call the *unitarization* in what follows. In short, for an operad  $\mathcal{P} \in \mathcal{O}p$ , the operad  $\tau\mathcal{P}$  is given by setting  $\tau\mathcal{P}(0) = *$  and by taking the quotient of the spaces  $\mathcal{P}(r)$  under relations that identify the composites  $p \circ_i e$ , where  $p \in \mathcal{P}(r+1)$  and  $e \in \mathcal{P}(0)$ , to the same point  $p \circ_i *$  in  $\tau\mathcal{P}(r)$ .

These functors define a Quillen adjunction  $\tau : \mathcal{O}p \rightleftarrows \mathcal{O}p_* : \iota$  where we provide  $\mathcal{O}p$  with the projective model structure (see for instance [2]), whereas we can provide  $\mathcal{O}p_*$  with the Reedy model structure, transported from the category  $\Lambda\mathcal{O}p_\emptyset$  (see [7, §II.8.4]), or with the projective model structure equally (as in [2]). The observation that our functors form a Quillen adjunction readily follows from the definition of the generating (acyclic) cofibrations of the category  $\mathcal{O}p$  as morphisms of free operads associated to generating (acyclic) cofibrations of the projective model category of symmetric sequences (see for instance [7, §II.8] for this background).

Both model categories are used to perform homotopy computations. To be specific, the authors use the model category  $\mathcal{O}p_*$  to compute mapping spaces of  $E_n$ -operads in [8], whereas the model category  $\mathcal{O}p$  is used in the Goodwillie-Weiss calculus, in the expression of the relationship between the mapping spaces of  $E_n$ -operads and the embedding spaces of Euclidean spaces with compact support (see for instance [5, 6, 9]). The approach of [8] is to use an operadic enhancement of the Sullivan model, which can be handled in the category  $\mathcal{O}p_*$  (see [7, §II.12]), in order to perform rational homotopy computations in the category of operads. The main purpose of this paper is to establish the following comparison statement, where we consider the (derived) mapping spaces  $\mathrm{Map}_{\mathcal{C}}^h(-, -)$  associated to these model categories  $\mathcal{C} = \mathcal{O}p, \mathcal{O}p_*$ :

**Theorem 1.** *We have a weak-equivalence of simplicial sets*

$$\mathrm{Map}_{\mathcal{O}p}^h(\iota\mathcal{P}, \iota\mathcal{Q}) \sim \mathrm{Map}_{\mathcal{O}p_*}^h(\mathcal{P}, \mathcal{Q}),$$

for all operads  $\mathcal{P}, \mathcal{Q} \in \mathcal{O}p_*$ .

Let  $\mathcal{P}, \mathcal{Q} \in \mathcal{O}p_*$  be as in this theorem. We pick a cofibrant resolution  $\mathcal{R}$  of the operad  $\iota\mathcal{P}$  in the category  $\mathcal{O}p$  and we assume that  $\mathcal{Q}$  is fibrant in  $\mathcal{O}p_*$  (if we consider the Reedy model structure). We then have

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$\text{Map}_{\mathcal{O}_p}^h(\iota P, \iota Q) = \text{Map}_{\mathcal{O}_p}(R, \iota Q)$  by definition of mapping spaces. We also have by adjunction  $\text{Map}_{\mathcal{O}_p}(R, \iota Q) \cong \text{Map}_{\mathcal{O}_{p_*}}(\tau R, Q)$ . We can therefore reduce the proof of Theorem 1 to the verification that we have the relation  $\tau R \xrightarrow{\sim} P$ , for any well-chosen cofibrant resolution  $R \xrightarrow{\sim} P$ , since  $\tau R$  is cofibrant in  $\mathcal{O}_{p_*}$  by Quillen adjunction, and since we have the implication  $\tau R \xrightarrow{\sim} P \Rightarrow \text{Map}_{\mathcal{O}_{p_*}}^h(P, Q) = \text{Map}_{\mathcal{O}_{p_*}}(\tau R, Q)$  in this case (by definition of mapping spaces again).

We can assume that  $P$  is the geometric realization of an operad in simplicial sets, which we abusively denote by the same letter  $P$ , because the model category of operads in simplicial sets is Quillen equivalent to the model category of topological operads (see for instance [7, §II.1.4]). We consider an arity-wise cartesian product  $E \times P$ , where  $E$  is an  $E_\infty$ -operad in simplicial sets, to get an operad in simplicial sets equipped with a free action of the symmetric group and such that  $E \times P \xrightarrow{\sim} P$ . (We make our choice of this operad  $E$  explicit later on.) We take  $R = W(E \times P)$ , the Boardman-Vogt construction on this operad  $E \times P$ , as a cofibrant resolution of  $P$ . We actually check that we have the following statement:

**Theorem 2.** *We have  $\tau W(E \times P) \xrightarrow{\sim} E \times P \xrightarrow{\sim} P$ .*

Thus, if we recap our arguments, then we can reduce the proof of Theorem 1 to the verification of this claim, and we devote the next section to this objective. To complete our results, we establish an analogue of our main theorems for the categories of  $k$ -truncated operads, considered in the paper [8], whose objects are operads  $P$  that are defined up to arity  $k$ . We check that our argument lines can be adapted to cover this case. We devote a second section to this survey. We use mapping spaces of  $k$ -truncated operads for the study of the Goodwillie-Weiss tower of embedding spaces in [8], and the  $k$ -truncated refinement of our comparison result is involved in such applications.

## 1. PROOF OF THE MAIN STATEMENTS

The goal of this section is to prove Theorem 2, and as a follow-up Theorem 1, as we just explained in the paper introduction. We make explicit our choice of the  $E_\infty$ -operad  $E$  first. We review the definition of the  $W$ -construction afterwards and we eventually give this proof of Theorem 2.

**1.1. The extended Barratt-Eccles operad.** We use the notation  $\Sigma_r$  for the symmetric group in  $r$  letters all along this paper, for any  $r \in \mathbb{N}$ .

The operad  $E$  which we consider in our construction is a simple extension of the classical Barratt-Eccles operad [1]. We define this operad as the classifying space  $E = B(\mathcal{M})$  of a certain operad in the category of categories  $\mathcal{M}$ . We first take:

$$(1) \quad \text{Ob } \mathcal{M} = \mathcal{F}(\mu(x_1, x_2), \mu(x_2, x_1), e) / \langle \mu \circ_1 \mu \equiv \mu \circ_2 \mu, \mu(e, e) \equiv e \rangle,$$

the operad in sets generated by a non-symmetric operation  $\mu = \mu(x_1, x_2)$  in arity 2, and an operation  $e$  in arity 0, together with the associativity relation  $\mu(\mu(x_1, x_2), x_3) \equiv \mu(x_1, \mu(x_2, x_3))$  and the idempotence relation  $\mu(e, e) = e$  as generating relations. We then set:

$$(2) \quad \text{Mor}_{\mathcal{M}(r)}(p, q) = pt,$$

for any pair of elements  $p, q \in \text{Ob } \mathcal{M}(r)$ . We define the composition operations of this operad  $\circ_i : \mathcal{M}(k) \times \mathcal{M}(l) \rightarrow \mathcal{M}(k+l-1)$  by the natural composition operations of the operad  $\text{Ob } \mathcal{M}$  at the object set level, and by the obvious identity maps at the morphism set level. (We give brief explanations on an interpretation of this operad in a remark at the end of the paper.)

If we use classical algebraic notation for the product operation  $x_1 x_2 = \mu(x_1, x_2)$ , then we can identify the elements of  $\text{Ob } \mathcal{M}(r)$  with monomials of the form:

$$(3) \quad p(x_1, \dots, x_r) = e^{\epsilon_0} x_{\sigma(1)} e^{\epsilon_1} \dots e^{\epsilon_{r-1}} x_{\sigma(r)} e^{\epsilon_r},$$

where  $\epsilon_0, \dots, \epsilon_r \in \{0, 1\}$ , and  $\sigma \in \Sigma_r$ . The operadic composition operations are given, at the object-set level, by the standard substitution operation of monomials together with the Boolean multiplication rules:

$$(4) \quad e^\alpha e^\beta = \begin{cases} e^0, & \text{if } \alpha = \beta = 0, \\ e^1, & \text{otherwise.} \end{cases}$$

The constant maps  $\mathcal{M}(r) \rightarrow pt$  trivially define equivalences of categories in all arities  $r \in \mathbb{N}$ , and as a consequence, these maps induce a weak-equivalence of operads in simplicial sets when we pass to classifying spaces:

$$(5) \quad \mathbf{E} = \mathbf{B}(\mathcal{M}) \xrightarrow{\sim} pt.$$

Let  $\overline{\mathbf{E}}_n = \overline{\mathbf{B}(\mathcal{M})}_n$  denote the collection of sets, where  $n \in \mathbb{N}$  is a fixed simplicial dimension, which we form by dropping (the degeneracies of) the vertex  $1 \in \mathbf{Ob} \mathcal{M}(1)$  (the operadic unit) from  $\overline{\mathbf{B}(\mathcal{M})}_n$ . The sets  $\overline{\mathbf{B}(\mathcal{M})}(1)_n$ ,  $n \in \mathbb{N}$ , are not preserved by the face operators of the classifying space  $\mathbf{B}(\mathcal{M})(1)$ . For instance, if we take the morphism  $x_1 e \rightarrow x_1$ , which represents a one simplex in  $\mathbf{B}(\mathcal{M})(1)$ , then we have  $d_0(x_1 e \rightarrow x_1) = x_1 = 1$ . But, on the other hand, we have the following lemma which makes this operad  $\mathbf{E} = \mathbf{B}(\mathcal{M})$  useful for our purpose:

**Lemma 1.** *The collections  $\overline{\mathbf{E}}_n = \overline{\mathbf{B}(\mathcal{M})}_n$ ,  $n \in \mathbb{N}$ , are preserved by the operadic composition operations of the operads in sets  $\mathbf{B}(\mathcal{M})_n$ , and hence, form operads without unit.*

*Proof.* The algebraic description of the object-set operad  $\mathbf{Ob} \mathcal{M}$  shows that this property holds for the collection of vertex sets  $\overline{\mathbf{Ob} \mathcal{M}} = \overline{\mathbf{B}(\mathcal{M})}_0$ . The conclusion of the lemma follows.  $\square$

**1.2. The  $W$ -construction and its unitarization.** We assume that  $\mathbf{P}$  is an operad in topological spaces for the moment. Briefly recall that the spaces  $W(\mathbf{P})(r)$  underlying the  $W$ -construction  $W(\mathbf{P})$  consist of collections  $[T; p_x, x \in VT; l_e, e \in \mathring{ET}]$ , where  $T$  is an  $r$ -tree (a tree with  $r$  ingoing edges numbered from 1 to  $r$ ), while  $p_x \in \mathbf{P}(r_x)$  is an operation associated to each vertex  $x \in VT$  (whose number of ingoing edges is denoted by  $r_x$ ), and  $l_e \in [0, 1]$  is a length associated to each inner edge  $e \in \mathring{ET}$  (see [3, 4]). To represent such a collection, we generally use a decoration of the tree  $T$ , with the elements  $p_x \in \mathbf{P}(r_x)$  on the vertices  $x \in VT$ , and the parameters  $l_e \in [0, 1]$  on the inner edges  $e \in \mathring{ET}$ . If we have  $l_\alpha = 0$  for some internal edge  $\alpha \in \mathring{ET}$  with  $v$  as target vertex and  $w$  as source vertex, then we have the relation:

$$(6) \quad \begin{array}{c} \dots \\ \diagup \quad \diagdown \\ \dots \quad p_v \\ \diagup \quad \diagdown \\ p_u \quad l_\alpha=0 \\ | \\ \dots \end{array} \equiv \begin{array}{c} \dots \quad \dots \\ \diagup \quad \diagdown \\ p_u \circ_\alpha p_v \\ | \\ \dots \end{array},$$

where we contract the edge  $\alpha$  in  $T$ , and we perform the composition operation  $p_u = p_v \circ_\alpha p_w$  to get an element of  $W(\mathbf{P})$  shaped on the tree  $T/\alpha$ . In  $W(\mathbf{P})$ , we also implement the relation:

$$(7) \quad \begin{array}{c} \dots \\ l_\beta | \\ 1 \\ l_\alpha | \\ \dots \end{array} \equiv \begin{array}{c} \dots \\ | \max(l_\alpha, l_\beta) \\ \dots \end{array}$$

when we have a vertex labeled by the unit element of the operad  $1 \in \mathbf{P}(1)$ . The operadic composite  $\circ_i$  of elements shaped on decorated trees  $S$  and  $T$  in  $W(\mathbf{P})$  is obtained by plugging the outgoing edge of the tree  $T$  in the  $i$ th ingoing edge of  $S$ , and by assigning the length  $l = 1$  to this new inner edge of the composite tree  $S \circ_i T$ . Recall that we have a weak-equivalence  $W(\mathbf{P}) \xrightarrow{\sim} \mathbf{P}$ , defined by forgetting about the length of the edges and by performing the composition operations shaped on our trees in the operad  $\mathbf{P}$ .

In  $\tau W(\mathbf{P})$ , we implement the extra reduction relation

$$(8) \quad \begin{array}{c} * \quad \dots \quad * \\ \diagdown \quad \diagup \\ S \\ |_{l_0=1} \\ \dots \end{array} \equiv \begin{array}{c} * \\ |_{l=1} \\ \dots \end{array},$$

when we have a whole subtree  $S$  with an outgoing edge of length  $l_0 = 1$  in which all chains of edges abut to an element of arity zero  $* \in P(0)$ .

**1.3. The cofibrant structure of the  $W$ -construction and the reduction to a non-unital  $W$ -construction.** We already mentioned that the model category of operads in simplicial sets is Quillen equivalent to the model category of operads in topological spaces. Therefore, we now assume (without loss of generality in our statement) that  $P$  is an operad in simplicial sets, of which we can take the geometric realization  $|P|$  to pass to the category of operads in topological spaces. Then we consider the operad  $E \times P$  such that  $(E \times P)(r) = E(r) \times P(r)$ , for any  $r \in \mathbb{N}$ . We can still take the geometric realization of this operad to get an operad in topological spaces  $|E \times P|$ . We then have  $E \sim pt \Rightarrow E \times P \sim P \Rightarrow |E \times P| \sim |P|$ . We moreover get that the  $W$ -construction of this operad  $W(|E \times P|)$  is cofibrant as an operad in topological spaces, which is not the case of the operad  $W(|P|)$  in general (when the symmetric groups do not operate freely on the components of the operad  $P$ ). We review the proof of a simplicial counterpart of this statement in the next paragraphs.

We can actually stay at the simplicial set level for our study, because we have an identity  $W(|E \times P|) = |W(E \times P)|$ , where we consider a simplicial version of the  $W$ -construction  $W(E \times P)$  before passing to the geometric realization. In brief, we just use the 1-simplex  $\Delta^1$  instead of the interval  $[0, 1]$  to adapt the construction of the previous paragraph to the simplicial setting. We may just note that the unit reduction relation (Eq. 7) is given by the geometric realization of a simplicial map  $m : \Delta^1 \times \Delta^1 \rightarrow \Delta^1$ . We refer to [3] for more details on the definition of this simplicial  $W$ -construction.

We then get that the operad  $W(E \times P)$  is cofibrant as an operad in simplicial sets (which is not the case of the operad  $W(P)$  again when the symmetric groups do not operate freely on the components of the operad  $P$ ). To check this observation, we use that the operad  $W(E \times P)$  admits a free structure  $W(E \times P) = \mathcal{F}(\dot{W}(E \times P))$ , for some generating collection  $\dot{W}(E \times P) \subset W(E \times P)$  which is preserved by the degeneracy operators, but not by the face operators of the simplicial structure on  $W(E \times P)$  (we therefore say that our operad is quasi-free, rather than free, in what follows). In short, the symmetric sequence  $\dot{W}(E \times P)$  consists of decorated trees such that  $l_e \neq 1$ , for all inner edges  $e$ . The indecomposable factors of the operadic decomposition of a decorated tree in  $W(E \times P)$  are the subtrees obtained by cutting all inner edges of length  $l_e = 1$ . The components  $\dot{W}(E \times P)(r)$  of the symmetric sequence  $\dot{W}(E \times P)$  inherit a free action of the symmetric group (we use the free symmetric structure of the cartesian product  $E \times P$  at this point), and the quasi-free operad  $W(E \times P) = \mathcal{F}(\dot{W}(E \times P))$  is cofibrant under this condition (see for instance [7, Theorem II.8.2.] )

We can now regard the  $W$ -construction  $W(E \times P)$  as an operad in bisimplicial sets with one simplicial dimension inherited from the simplices  $\Delta^1$ , which we attach to the inner edges of our decorated trees, and the other simplicial dimension given by the internal simplicial grading of the operad  $E \times P$ . We then take the diagonal complex  $W(E \times P) = \text{Diag } W_\bullet((E \times P)_\bullet)$  to retrieve an operad in simplicial sets from this bisimplicial object.

We can obviously extend the result of Lemma 1 to the collections of sets  $(\overline{E \times P})_n$  which we form by dropping the unit object  $(1, 1) \in E(1) \times P(1)$  from the operads in sets  $(E \times P)_n = E_n \times P_n$ ,  $n \in \mathbb{N}$ . We then have the following easy consequence of this observation:

**Lemma 2.** *We have an identity  $W((E \times P)_n) = W'(\overline{E \times P}_n)$ , when we fix the internal simplicial dimension of the operad  $E \times P$ , where  $W'$  denotes a version of the  $W$ -construction for non-unital operads which we merely define by forgetting about the unit reduction relation (Eq. 7) of the standard construction.*  $\square$

We now assume  $P(0) = *$ . We note that the operad  $E$  satisfies the relation  $E(0) = *$  too, and as a consequence, so does the operad  $E \times P$ . We accordingly get that the augmentation of the  $W$ -construction  $W(E \times P) \rightarrow E \times P$  induces a morphism  $\tau W(E \times P) \rightarrow E \times P$  in the category  $\mathcal{Op}_*$  by adjunction. We have an obvious counterpart of the result of the previous lemma for the operad  $\tau W((E \times P)_n)$ . We use this observation in the verification of the following claim:

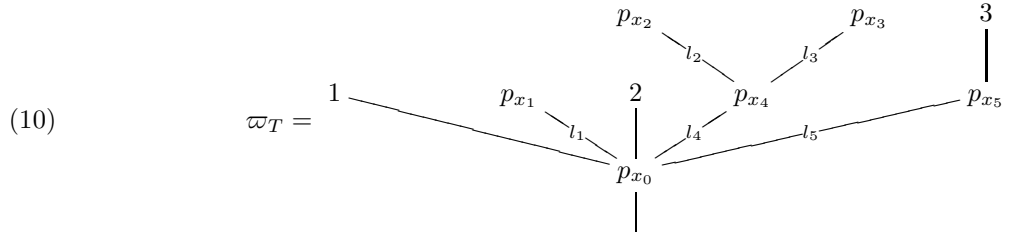
**Lemma 3.** *The augmentation map  $\tau W((E \times P)_n)(r) \rightarrow (E \times P)_n(r)$  defines a weak-equivalence of simplicial sets, for each dimension  $n \in \mathbb{N}$  and for any arity  $r \in \mathbb{N}$ , where we regard the set  $(E \times P)_n(r)$  as a discrete simplicial set.*

*Proof.* We set  $\mathbf{N}_n = (\overline{\mathbf{E} \times \mathbf{P}})_n$  for short and we use the identity  $\tau W((\mathbf{E} \times \mathbf{P})_n) = \tau W'(\mathbf{N}_n)$ . We aim to prove that we have a weak-equivalence of simplicial sets  $\epsilon : \tau W'(\mathbf{N}_n)(r) \xrightarrow{\sim} \mathbf{N}_n(r)$ , for each dimension  $n \in \mathbb{N}$  and for any arity  $r \in \mathbb{N}$ . We can take the geometric realization of this map  $\epsilon : |\tau W'(\mathbf{N}_n)(r)| \rightarrow \mathbf{N}_n(r)$  to establish this claim. We have an obvious map going the other way round  $\eta : \mathbf{N}_n(r) \rightarrow |\tau W'(\mathbf{N}_n)(r)|$ , which merely carries any element  $p \in \mathbf{N}_n(r)$  to an  $r$ -corolla with  $p$  as label in  $\tau W'(\mathbf{N}_n)(r)$ . We have  $\epsilon\eta = id$  and we are left to verifying the relation  $\eta\epsilon \simeq id$  in the homotopy category of spaces.

We proceed as follows. Let  $\varpi_T = [T; p_x, x \in VT; l_e, e \in \mathring{ET}]$  be a collection which represents a point of the cell complex  $|\tau W'(\mathbf{N}_n)(r)|$ . Instead of assigning a length to the internal edges  $l_e$ ,  $e \in \mathring{ET}$ , we can equivalently assign a height  $h_x \in [0, \infty[$  to each vertex  $x \in VT$ , with the following rules (Eq. 9), in order to parameterize the elements of our complex:

$$(9) \quad h_x = \begin{cases} 0, & \text{if } x \text{ is the source vertex of the outgoing edge of the tree (the root),} \\ h_y + l_e, & \text{if } x \text{ is a source vertex of an inner edge } e \in \mathring{ET} \text{ with } y \text{ as target vertex.} \end{cases}$$

For instance, in the case of the decorated 3-tree



with  $VT = \{x_0, x_1, x_2, x_3, x_4, x_5\}$ , we get  $h_{x_0} = 0$ ,  $h_{x_5} = l_5$ ,  $h_{x_4} = l_4$ ,  $h_{x_3} = l_3 + l_4$ ,  $h_{x_2} = l_2 + l_4$ , and  $h_{x_1} = l_1$ . In order to give a sense to this correspondence, we crucially use that no unit reduction relation occurs in  $|\tau W'(\mathbf{N}_n)(r)|$ . (The height functions would not be well-defined otherwise.)

We consider the continuous family  $\varpi_T^t \in |\tau W'(\mathbf{N}_n)(r)|$ ,  $t \in [0, \infty]$ , defined by making these height parameters vary by the formula:

$$(11) \quad h_x^t = \min(h_x, t),$$

for all  $x \in VT$ , with the obvious convention  $\min(h, \infty) = h$ . We readily check that the mapping  $(t, \varpi_T) \mapsto \varpi_T^t$  is compatible with the identification relations of the cell complex  $|\tau W'(\mathbf{N}_n)(r)|$ . We therefore get that this mapping  $(t, \varpi_T) \mapsto \varpi_T^t$  defines a continuous family of maps  $\rho_t : |\tau W'(\mathbf{N}_n)(r)| \rightarrow |\tau W'(\mathbf{N}_n)(r)|$  such that  $\rho_\infty = id$  and  $\rho_0 = \eta\epsilon$ , since making the assignment  $h_x^0 = \min(h_x, 0)$  amounts to assigning the length  $l_e = 0$  to all inner edges  $e \in \mathring{ET}$  of the tree  $T$ , and hence, to contracting these edges in  $|\tau W'(\mathbf{N}_n)(r)|$ .

We can compose the mapping  $t \mapsto \rho_t$  with the function  $t \mapsto t/(1-t)$  to retrieve a continuous family of maps defined for a value  $t \in [0, 1]$  of the time parameter  $t$ , as in the usual definition of a homotopy. Note that we have  $h_x^t \equiv h_x$  for  $t \gg h$ , for a bound  $h$  that only depends on the tree superstructure  $T$ , so that no continuity problem occurs at  $t = \infty$  in our construction.  $\square$

We can now complete the:

*Proof of Theorem 2.* We use the general statement that a horizontal weak-equivalence of bisimplicial sets  $\phi : X_{\bullet, n} \xrightarrow{\sim} Y_{\bullet, n}$ ,  $n \in \mathbb{N}$ , induces a weak-equivalence when we pass to the diagonal complex  $\phi : \text{Diag } X_{\bullet, \bullet} \xrightarrow{\sim} \text{Diag } Y_{\bullet, \bullet}$  to conclude that the weak-equivalences of the previous lemma  $\tau W((\mathbf{E} \times \mathbf{P})_n)(r) \xrightarrow{\sim} (\mathbf{E} \times \mathbf{P})_n(r)$ ,  $n \in \mathbb{N}$ , induce a weak-equivalence of simplicial sets  $\tau W(\mathbf{E} \times \mathbf{P})(r) \xrightarrow{\sim} (\mathbf{E} \times \mathbf{P})(r)$ , for each arity  $r \in \mathbb{N}$ , which is nothing but the claim of Theorem 2.  $\square$

Recall that Theorem 1 is a corollary of Theorem 2. Thus, the previous verification also completes the proof of Theorem 1.  $\square$

**1.4. Remark.** In §1.1, we can identify the categories of algebras associated to the operad  $\mathcal{M}$  with the category of categories  $\mathcal{C}$  equipped with a strictly associative tensor product  $\otimes : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$  and an object  $e \in \mathcal{C}$ , which is strictly idempotent  $e \otimes e = e$ , but which only satisfies the unit relations of tensor products up to natural isomorphisms in general  $x \otimes e \simeq x \simeq e \otimes x$  (compare with the statement of [7, Theorem I.6.3.2-6.3.3]). The tensor product  $\otimes : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$  represents the image of the generating operation  $\mu \in \mathbf{Ob} \mathcal{M}(2)$  under the functor  $\phi : \mathcal{M} \rightarrow \mathbf{End}_{\mathcal{C}}$  that encodes the action of the operad  $\mathcal{M}$  on  $\mathcal{C}$ , where  $\mathbf{End}_{\mathcal{C}}$  denotes the endomorphism operad of  $\mathcal{C}$  in the category of categories. The object  $e \in \mathcal{C}$  represents the image of the zero-ary operation  $e \in \mathbf{Ob} \mathcal{M}(0)$ . The natural transformations  $x \otimes e \simeq x \simeq e \otimes x$ , for  $x \in \mathcal{C}$ , are given by the image of the corresponding isomorphisms  $x_1 e \simeq x_1 \simeq e x_1$  in the morphism sets of the category  $\mathcal{M}(1)$ . (We refer to [7, §I.6.3] for a detailed proof of several variants of this correspondence.)

## 2. THE CASE OF $k$ -TRUNCATED OPERADS

We now examine the proof of the counterpart of our main statements for  $k$ -truncated operads. We mainly briefly check that our constructions and argument lines go through (or can be adapted) in this setting. We recall the definition of the category of  $k$ -truncated operads first. We review the proof of our main results afterwards.

Briefly say that we call “ $k$ -truncated operad” the structure formed by an operad  $\mathbf{P}$  whose components  $\mathbf{P}(r)$  are only defined for  $r \leq k$ , and where we restrict ourselves to composition products  $\circ_i : \mathbf{P}(m) \times \mathbf{P}(n) \rightarrow \mathbf{P}(m+n-1)$  that preserve this arity bound. This condition is equivalent to the relation  $m+n-1 \leq k$  in the case  $n > 0$ , and to the relation  $m \leq k$  in the case  $n = 0$ . In the definition of a  $k$ -truncated operad, we also restrict the application of the associativity relation of operads to the cases where the composition products involved in the relation preserve the arity bound, so that our relation makes sense.

We use the notation  $\mathcal{Op}^{\leq k}$  for this category of  $k$ -truncated operads, and the notation  $\mathcal{Op}_*^{\leq k} \subset \mathcal{Op}^{\leq k}$  for the associated subcategory of unitary operads, for which we assume  $\mathbf{P}(0) = *$ . We can also adapt the definition of the notion of a  $\Lambda$ -operad to the  $k$ -truncated setting. We then consider operads equipped with restriction operators  $u^* : \mathbf{P}(n) \rightarrow \mathbf{P}(m)$  defined for all injective maps  $u : \{1 < \dots < m\} \rightarrow \{1 < \dots < n\}$  such that  $1 \leq m \leq n \leq k$ . We also restrict ourselves to the cases where the arity bound is preserved in the expression of equivariance of composition products  $\circ_i$  with respect to these restriction operators in [7, Proposition I.2.2.16]. We use the notation  $\Lambda \mathcal{Op}_{\vec{\partial}}^{\leq k}$  for this  $k$ -truncated analogue of the category of  $\Lambda$ -operads. We again have an isomorphism of categories  $\mathcal{Op}_*^{\leq k} \cong \Lambda \mathcal{Op}_{\vec{\partial}}^{\leq k}$ . We mostly deal with the category  $\mathcal{Op}_*^{\leq k}$  (rather than  $\Lambda \mathcal{Op}_{\vec{\partial}}^{\leq k}$ ) in what follows.

The definition of free operads has an obvious counterpart in the  $k$ -truncated context, which gives a left adjoint  $\mathcal{F} : \mathcal{Seq}^{\leq k} \rightarrow \mathcal{Op}^{\leq k}$  of the obvious forgetful functor  $\omega : \mathcal{Op}^{\leq k} \rightarrow \mathcal{Seq}^{\leq k}$  from the category of  $k$ -truncated operads  $\mathcal{Op}^{\leq k}$  to the category of  $k$ -truncated symmetric sequences  $\mathcal{Seq}^{\leq k}$  (the category of symmetric sequences  $\mathcal{M}$  with components  $\mathcal{M}(r)$  defined for  $r \leq k$ ). Recall that the ordinary free operad  $\mathcal{F}(\mathcal{M})$  generated by a symmetric sequence  $\mathcal{M}$  consists of decorated trees  $T$  with  $r$ -ingoing edges, numbered from 1 to  $r$ , and whose vertices  $x$  are labeled by elements  $\xi_x \in \mathcal{M}(r_x)$  of the symmetric sequence  $\mathcal{M}$ , where we again use the notation  $r_x$  to denote the number of ingoing edges of our vertex  $x$  in the tree  $T$ . In the case of  $k$ -truncated operads, we just restrict ourselves to the case  $r \leq k$ , and we assume  $r_x \leq k$  for all vertices of our trees  $x \in VT$ . The adjunction morphism  $\lambda : \mathcal{F}(\mathbf{P}) \rightarrow \mathbf{P}$  carries any such decorated tree with  $\mathcal{M} = \mathbf{P}$  to a corresponding treewise composite operation in the operad  $\mathbf{P}$ . Let us mention that some care is necessary in the context of  $k$ -truncated operads since some intermediate composites which we may form by contracting subtrees in a treewise tensor product (see [7, §A.2.5]) go beyond our arity bound. To avoid this problem, we can evaluate all composites with factors of arity zero at first. Then we obtain a treewise tensor product shaped on a tree with  $r \leq k$  ingoing edges and in which all indecomposable factors have a positive arity  $r_x > 0$ . This property ensures that all partial composites which we may form inside our treewise tensor product do not go over our arity bound.

We can adapt the definition of the projective model structure of operads (in simplicial sets, in topological spaces) to the category of  $k$ -truncated operads  $\mathcal{Op}^{\leq k}$ . We again assume that a morphism of  $k$ -truncated operads is a weak-equivalence (respectively, a fibration) if this morphism forms a weak-equivalence (respectively, a fibration) in the base category (of simplicial sets, of topological spaces) arity-wise, and we characterize

the cofibrations by the left lifting property with respect to the class of acyclic fibrations. We take the morphisms of free objects  $\mathcal{F}(i) : \mathcal{F}(\mathcal{M}) \rightarrow \mathcal{F}(\mathcal{N})$  induced by generating (acyclic) cofibrations of the category of  $k$ -truncated symmetric sequences as a set of generating (acyclic) cofibrations in this model category  $\mathcal{O}p^{\leq k}$ . We can also adapt the definition of the Reedy model category of  $\Lambda$ -operads  $\Lambda \mathcal{O}p_{\varnothing}$  in [7, §II.8.4] to the category of  $k$ -truncated  $\Lambda$ -operads  $\Lambda \mathcal{O}p_{\varnothing}^{\leq k}$ . We again use the isomorphism of categories  $\mathcal{O}p_{\ast}^{\leq k} \cong \Lambda \mathcal{O}p_{\varnothing}^{\leq k}$  to transport this model structure to the category of  $k$ -truncated unitary operads  $\mathcal{O}p_{\ast}^{\leq k}$ . For our purpose, we can equivalently equip this category  $\mathcal{O}p_{\ast}^{\leq k}$  with a restriction of the projective model structure on  $\mathcal{O}p^{\leq k}$ .

We can still form a Quillen adjunction  $\tau : \mathcal{O}p^{\leq k} \rightleftarrows \mathcal{O}p_{\ast}^{\leq k} : \iota$  by taking the canonical category embedding  $\iota : \mathcal{O}p_{\ast}^{\leq k} \hookrightarrow \mathcal{O}p^{\leq k}$  on the one hand and the obvious  $k$ -truncated counterpart of our unitarization functor  $\tau : \mathcal{O}p^{\leq k} \rightarrow \mathcal{O}p_{\ast}^{\leq k}$  on the other hand. We then have the following counterpart of the result of Theorem 1:

**Theorem 1'.** *We have a weak-equivalence of simplicial sets*

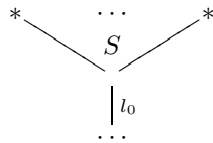
$$\mathrm{Map}_{\mathcal{O}p^{\leq k}}^h(\iota P, \iota Q) \sim \mathrm{Map}_{\mathcal{O}p_{\ast}^{\leq k}}^h(P, Q),$$

for all operads  $P, Q \in \mathcal{O}p_{\ast}^{\leq k}$ .

We still reduce the proof of this statement to the verification that we have a weak-equivalence  $\tau R \xrightarrow{\sim} P$ , for a cofibrant resolution  $R \xrightarrow{\sim} \iota P$  of the object  $\iota P$  in the category of all  $k$ -truncated operads  $\mathcal{O}p$ .

We take  $R = W^{\leq k}(\mathbf{E} \times \mathbf{P})$ , where  $W^{\leq k}(-)$  denotes a  $k$ -truncated analogue of the  $W$ -construction, and  $\mathbf{E} \times \mathbf{P}$  denotes the  $k$ -truncated operad such that  $(\mathbf{E} \times \mathbf{P})(r) = \mathbf{E}(r) \times \mathbf{P}(r)$ , for  $r \leq k$ , with  $\mathbf{E}$  defined as in §1.1. We use the same construction as in §§1.2-1.3 to define this  $k$ -truncated version of the  $W$ -construction (in both the topological setting and the simplicial setting). We just restrict ourselves to decorated trees such that, by cutting any subset of inner edges that contains the edges of length  $l_e = 1$ , we get a decomposition of our tree into subtrees  $S$  with at most  $k$  ingoing edges each. In particular, we assume that the vertices of our trees  $x$  have at most  $k$  ingoing edges each, so that the corresponding labels  $p_x \in \mathbf{P}(r_x)$  satisfy our arity bound  $r_x \leq k$ . Our condition also ensures that the edge contraction relations in the definition of our object (Eq. 6) produce allowable composition operations in our  $k$ -truncated operad  $\mathbf{P}$ , and that  $R = W^{\leq k}(\mathbf{E} \times \mathbf{P})$  forms a quasi-free object in the category of  $k$ -truncated operads  $W^{\leq k}(\mathbf{E} \times \mathbf{P}) = \mathcal{F}(\dot{W}^{\leq k}(\mathbf{E} \times \mathbf{P}))$  with the same definition as in §1.3 for the generating symmetric sequence  $\dot{W}^{\leq k}(\mathbf{E} \times \mathbf{P})$  (our condition implies that the treewise tensor products that form this symmetric sequence have an arity  $r \leq k$ ). We still deduce from this observation that  $W^{\leq k}(\mathbf{E} \times \mathbf{P})$  forms a cofibrant object in the category of  $k$ -truncated operads.

We moreover have a canonical morphism  $W^{\leq k}(\mathbf{E} \times \mathbf{P}) \rightarrow \mathbf{E} \times \mathbf{P}$ , which we obtain by contracting the edges of our decorated trees and by performing the corresponding composites in  $\mathbf{E} \times \mathbf{P}$ . We can still prove that this morphism is a weak-equivalence, but some care is needed there, since we have to adapt our construction in order to ensure that our contracting homotopy produces decorated trees that fulfill the arity bound conditions of the operad  $W^{\leq k}(\mathbf{E} \times \mathbf{P})$ . We can proceed in two steps. In a first step, we can apply the contracting homotopy of the proof of Lemma 3 to the maximal subtrees of the form



in a decorated tree, where we now use the notation  $*$  for any arity zero element of our operad  $\mathbf{P}$  (we do not necessarily assume that  $\mathbf{P}$  belongs to the subcategory  $\mathcal{O}p_{\ast}$  for the moment). For instance, in the case of the tree of Eq. 10, we consider the subtrees:

$$S_1 = \begin{array}{c} p_{x_1} \\ | \\ l_1 \\ | \\ \vdots \end{array} \quad \text{and} \quad S_2 = \begin{array}{c} p_{x_2} \quad p_{x_3} \\ \quad \diagdown \quad \diagup \\ \quad \quad l_2 \quad l_3 \\ \quad \quad \quad \diagdown \quad \diagup \\ \quad \quad \quad \quad p_{x_4} \\ \quad \quad \quad \quad | \\ \quad \quad \quad \quad l_4 \\ \quad \quad \quad \quad \vdots \end{array}.$$

This operation has the effect of reducing the composites with operations of arity zero in our object and of eliminating the vertices with no ingoing edge. Thus, as a result of this first contracting homotopy operation,

we get a decorated tree with  $r \leq k$  ingoing edges and of which all vertices  $x \in VT$  have  $r_x > 0$  ingoing edges. Then we can apply our contracting homotopy a second time, to our whole decorated tree this time, in order to abut to a corolla, which corresponds to an element of the operad  $\mathbf{E} \times \mathbf{P}$  inside  $W^{\leq k}(\mathbf{E} \times \mathbf{P})$ . The condition  $r_x > 0$  ensures that all intermediate composites which we form in this second retraction process fulfill our arity bound and define allowable elements of the  $k$ -truncated  $W$ -construction. Thus, we eventually conclude that our morphism  $W^{\leq k}(\mathbf{E} \times \mathbf{P}) \rightarrow \mathbf{E} \times \mathbf{P}$  forms a homotopy equivalence arity-wise, and hence, defines a weak-equivalence in the category of  $k$ -truncated operads, like the canonical projection  $\mathbf{E} \times \mathbf{P} \rightarrow \mathbf{P}$ , so that  $W^{\leq k}(\mathbf{E} \times \mathbf{P})$  forms a cofibrant resolution of our object  $\mathbf{P}$  in this category  $\mathcal{O}p^{\leq k}$ .

We now assume that  $\mathbf{P}$  is a unitary  $k$ -truncated operads in the category of simplicial sets, so that  $\mathbf{P}(0) = *$ . We can readily adapt the observation of Lemma 2 in the  $k$ -truncated context, and we can also use the above two-steps process to adapt the definition of the contracting homotopy of the proof of Lemma 3 to the case of the unitarization of the  $k$ -truncated  $W$ -construction  $\tau W^{\leq k}(\mathbf{E} \times \mathbf{P})$ . We therefore have the following statement:

**Theorem 2’.** *We have  $\tau W^{\leq k}(\mathbf{E} \times \mathbf{P}) \xrightarrow{\sim} \mathbf{E} \times \mathbf{P} \xrightarrow{\sim} \mathbf{P}$ , for all  $k$ -truncated unitary operads  $\mathbf{P} \in \mathcal{O}p_*^{\leq k}$ .*  $\square$

Recall again that this theorem gives the result of Theorem 1’ when we take  $\mathbf{R} = W^{\leq k}(\mathbf{E} \times \mathbf{P})$  as a cofibrant resolution of the  $k$ -truncated operad  $\mathbf{P} \in \mathcal{O}p_*^{\leq k}$  to compute our derived mapping spaces. Thus, the proof of this statement, Theorem 1’, is complete.  $\square$

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